RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

E. DI BENEDETTO
R. SPIGLER

An algorithm for the one-phase Stefan problem

Rendiconti del Seminario Matematico della Università di Padova, tome 69 (1983), p. 109-134
<http://www.numdam.org/item?id=RSMUP_1983__69__109_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 69 (1983)

An Algorithm for the One-Phase Stefan Problem

E. DI BENEDETTO - R. SPIGLER (*)

1. Introduction.

Consider the following one-phase one-dimensional Stefan problem.

$$(\mathcal{S}\mathcal{T}) \quad \begin{cases} Lu \equiv u_{xx} - u_t = 0 & \text{in} \quad D_T \equiv \{0 < x < s(t)\} \times (0, T] \\ u(x, 0) \quad = h(x)\,, & 0 < x \leqslant s(0) = b\,,\ (b > 0) \\ u_x(0, t) \quad = g(u(0, t), t)\,, & 0 < t \leqslant T \\ u(s(t), t) = 0\,, & 0 < t \leqslant T \\ u_x(s(t), t) = -\dot{s}(t)\,, & 0 < t \leqslant T \end{cases}$$

where $x \to h(x)$, $(\xi, t) \to g(\xi, t)$ are given functions on $(0, b]$ and $R \times (0, T]$ respectively.

Under suitable assumptions on $h(\cdot)$ and g, $(\mathcal{S}\mathcal{T})$ admits a unique classical solution. For such results we refer to the survey article [13] and other papers given in the extensive bibliography.

The aim of this paper is to propose an algorithm to construct the solution, which consists in solving the heat equation in progressively increasing rectangles, whose size is controlled by the Stefan condition $u_x(s(t), t) = -\dot{s}(t)$.

Such an algorithm arises as a natural modification of Hüber's

(*) Indirizzo degli AA.: E. DIBENEDETTO: Indiana University, Bloomington, IN 47405; R. SPIGLER: Istituto di Matematica Applicata, Università di Padova, Italy.

Sponsored by the United States Army under Contract no. DAAG 29-80-C-0041.

method [15, 10, 1] and can be described in a simple fashion as follows.

First the interval $[0, T]$ is divided in n intervals of length $\theta = T/n$, then for $t \in [0, \theta]$ we set $s_\theta(t) = b$ and solve the problem

$$(\mathcal{T}_1) \qquad \begin{cases} u^1_{xx} - u^1_t = 0 \quad \text{in} \quad R_1 \equiv \{0 < x < b\} \times (0, \theta] \\ u^1_x(0, t) \;= g(u^1(0, t), t)\,, \qquad 0 < t \leqslant \theta \\ u^1(x, 0) \;= h(x)\,, \qquad 0 < x \leqslant b \\ u^1(b, t) \;= 0\,, \qquad 0 < t \leqslant \theta\,. \end{cases}$$

We compute the number $u^1_x(s_\theta(\theta) = b, \theta)$ and determine the rectangle

$$R_2 \equiv \{0 < x < x_2 = b - u^1_x(b, \theta)\theta\} \times (\theta, 2\theta]\,,$$

setting $s_\theta(t) = x_2$ for $t \in (\theta, 2\theta]$. In R_2 we solve a problem similar to $(\mathcal{T}_1)$, and proceed in this fashion.

The convergence of schemes where at each time step the free boundary is approximated by a vertical segment was conjectured by Datzeff [5, 6].

A proof of convergence has been given by Fasano-Primicerio-Fontanella [11]. Their scheme however is somewhat more complicated than the one we propose here, both in the construction of the sequence of rectangles and in the boundary conditions on x_j, $(j-1)\theta < t \leqslant j\theta$ which are not homogeneous, being given as a relationship linking the distance $t - (j-1)\theta$ with the values $u^{j-1}_x(x_{j-1}, (j-1)\theta)$.

Thus the scheme we have described has a two-fold simplicity: the rectangular geometry and the homogeneity of the boundary data on the approximating free boundary.

We treat the problem for boundary data on $x = 0$ of variational type since such a condition is the « natural » one, as pointed out and discussed in [7]. The method could handle the Dirichlet boundary data as well.

We give an estimate of the speed of convergence which turns out to be of the order of $\sqrt{\theta}$.

As a related work we mention the methods of [14] based on enthalpy considerations, which yield a rate of convergence of the order of $[\theta \ln \theta^{-1}]^{\frac{1}{2}}$.

The methods of proof are simple in that we exploit both the rectangular geometry and the homogeneity of the data to represent the approximating solutions by means of elementary heat potentials.

Section 2 contains the precise description of the algorithm, assumptions and statement of results. In Section 3 and 4 we produce basic estimates and prove the convergence of the approximating solution, to the solution of $(S\mathcal{T})$.

The error estimate is given in Section 5. We conclude the paper by discussing some variants of the scheme.

2. Assumptions and statement of results.

Throughout the paper we will make the following assumptions on the data.

[A_1] $x \to h(x)$ is a positive Lipschitz continuous function on $[0, b]$, with Lipschitz constant H, $h(0) > 0$ and $h(b) = 0$.

[A_2] $(\xi, t) \to g(\xi, t)$ is non-positive on $R \times (0, T]$, continuous with respect to $t \in (0, T]$, Lipschitz continuous in ξ, uniformly in t, with Lipschitz constant G_1 and $g(h(0), 0) = h'(0)$. Moreover there exists a non-negative constant G_2 such that

$$|g(\xi, t)| \leqslant G_1|\xi| + G_2\,, \qquad (\xi, t) \in R \times (0, T]\,.$$

For $n = 1, 2, \ldots$, set $\theta = T/n$ and consider the sequence of problems $\mathcal{T}_j$, $j = 1, 2, \ldots, n$, defined by

$$(\mathcal{T}_j)\quad \begin{cases} Lu^j = 0 \quad \text{in} \quad R_j \equiv \{0 < x < x_j\} \times \{(j-1)\theta < t \leqslant j\theta\} \\ u^j(x, (j-1)\theta) = \begin{cases} u^{j-1}(x, (j-1)\theta)\,, & 0 < x \leqslant x_{j-1} \\ 0\,, & x_{j-1} < x \leqslant x_j \end{cases} \\ u^j_x(0, t) = g(u^j(0, t), t)\,, & (j-1)\theta < t \leqslant j\theta \\ u^j(x_j, t) = 0\,, & (j-1)\theta < t \leqslant j\theta \end{cases}$$

where the sequence $\{x_j\}_{j=1}^n$ is recursively defined by

$$x_0 = x_1 = b\,; \qquad x_j = x_{j-1} - u^{j-1}_x(x_{j-1}, (j-1)\theta)\cdot\theta\,, \qquad j = 2, 3, \ldots, n$$

and

$$u^0(x, 0) = u^1(x, 0) \equiv h(x) \qquad \text{in } (0, b]\,.$$

By virtue of $[A_1]$-$[A_2]$ each $(\mathcal{T}_j)$ admits recursively a unique classical solution u^j whose derivative u^j_x exists up to the lateral boundaries of R_j. Consequently the sequence $\{x_j\}$ is well defined.

Setting

$$s_\theta(0) = b\,; \qquad s_\theta(t) = x_j \qquad \text{for } (j-1)\theta < t \leqslant j\theta,\ j = 1, 2, \ldots, n,$$

we obtain a left-continuous, piecewise constant function defined in $[0, T]$. By elementary considerations and the maximum principle [7, 8] $u^j(x, t) \geqslant 0$ and the numbers $u^j_x(x_j, j\theta)$ are non-positive so that $x_{j+1} \geqslant x_j$ and $s_\theta(\cdot)$ is non-decreasing.

On the domain

$$\mathfrak{D}_{s_\theta} \equiv \bigcup_{j=1}^{n} R_j\,,$$

we define the function $(x, t) \to u_\theta(x, t)$, $(x, t) \in \mathfrak{D}_{s_\theta}$, by setting

$$u_\theta(x, t) = u^j(x, t)\,, \qquad (x, t) \in R_j\,, \qquad j = 1, 2, \ldots, n\,.$$

We will think of $u(\cdot, \cdot)$ the solution of $(\mathcal{S}\mathcal{T})$ and u_θ as defined in the whole half strip $S \equiv (0, \infty) \times (0, T]$, by setting them to be equal to zero outside D_T and $\mathfrak{D}_{s_\theta}$ respectively. We will use this device for the various functions appearing in what follows without specific mention.

For bounded functions $(x, t) \to w(x, t)$, $t \to f(t)$ defined in S and $(0, t]$ respectively we set

$$\|w\|_{\infty,S} = \sup_{(x,t)\in S} |w(x, t)|$$

$$\|f\|_t = \sup_{0<\tau\leqslant t} |f(\tau)|\,.$$

We can now state our main result.

THEOREM. As $\theta \to 0$, $u_\theta(x, t) \to u(x, t)$ uniformly in S and $s_\theta(t) \to s(t)$ uniformly in $[0, T]$. Moreover there exists a constant C depending upon H, b, G_1, G_2, T such that

$$\|u_\theta - u\|_{\infty,S} \leqslant C\sqrt{\theta}$$

$$\|s_\theta - s\|_T \leqslant C\sqrt{\theta}\,.$$

REMARK. (i) In view of the stability of $(\mathcal{S}\mathcal{T})$ (see [2, 3]) the Lipschitz condition in $[A_1]$ can be replaced by

$[A_1]'$ $\quad x \to h(x)$ is essentially bounded in $[0, b]$.

(ii) The signum condition on g in $[A_2]$ can be dropped, provided we assume $g(0, t) = 0$, $t \in (0, T]$. In this case we set $G_2 = 0$ in the growth condition for g.

3. Some basic estimates.

Let

$$\Gamma(x, t; \xi, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right]$$

be the fundamental solution of the heat equation and let $G(x, t; \xi, \tau)$, $N(x, t; \xi, \tau)$ be the Green and Neumann's functions respectively, defined by

$$G(x, t; \xi, \tau) = \Gamma(x, t; \xi, \tau) - \Gamma(-x, t; \xi, \tau)\,,$$

$$N(x, t; \xi, \tau) = \Gamma(x, t; \xi, \tau) + \Gamma(-x, t; \xi, \tau)\,.$$

In the j-th rectangle R_j the solution u^j of $(\mathcal{T}_j)$ can be implicitly represented as

$$(3.1)\qquad u^j(x, t) = \int_0^{x_j} N\big(x, t; \xi, (j-1)\theta\big)\, u^j\big(\xi, (j-1)\theta\big)\, d\xi - \\ - \int_{(j-1)\theta}^{t} N(x, t; 0, \tau)\, g\big(u^j(0, \tau), \tau\big)\, d\tau + \int_{(j-1)\theta}^{t} N(x, t; x_j, \tau)\, u^j_x(x_j, \tau)\, d\tau$$

for $(x, t) \in R_j$. Taking the derivative with respect to x in (3.1) and letting $x \to x_j$ we obtain

$$(3.2)\qquad \tfrac{1}{2} u^j_x(x_j, t) = \int_0^{x_j} G\big(x_j, t; \xi, (j-1)\theta\big)\, u^j_x\big(\xi, (j-1)\theta\big)\, d\xi -$$

$$-\int_{(j-1)\theta}^{t} N_x(x_j, t; 0, \tau)\, g\big(u^j(0, \tau), \tau\big)\, d\tau +$$

$$+\int_{(j-1)\theta}^{t} N_x(x_j, t; x_j, \tau)\, u_x^j(x_j, \tau)\, d\tau\,, \qquad (j-1)\,\theta < t \leqslant j\theta\,.$$

The calculations leading to (3.1)-(3.2) are routine and we refer to [4, 9, 12] for details.

Let us fix $1 < j \leqslant n$ and $(x, t) \in R_j$ and integrate the Green identity

$$\frac{\partial}{\partial \xi}\,(N u_\xi - u N_\xi) - \frac{\partial}{\partial \tau}\,(N u) = 0$$

over R_k, $1 \leqslant k < j$. Since in R_k we are away from the singularity we obtain

$$\text{(3.3)} \qquad \int_0^{x_k} N(x, t; \xi, k\theta)\, u^k(\xi, k\theta)\, d\xi -$$

$$-\int_0^{x_k} N\big(x, t; \xi, (k-1)\,\theta\big)\, u^k\big(\xi, (k-1)\,\theta\big)\, d\xi =$$

$$= -\int_{(k-1)\theta}^{k\theta} N(x, t; 0, \tau)\, g\big(u_\theta(0, \tau), \tau\big)\, d\tau + \int_{(k-1)\theta}^{k\theta} N(x, t; x_k, \tau)\, u_x^k(x_k, \tau)\, d\tau$$

for $k = 1, 2, \ldots, (j-1)$.

By virtue of our definition of $(\mathcal{T}_j)$, the second integral on the left hand side of (3.3) can be rewritten as

$$-\int_0^{x_{k-1}} N\big(x, t; \xi, (k-1)\,\theta\big)\, u^{k-1}\big(\xi, (k-1)\,\theta\big)\, d\xi\,.$$

Consequently adding the identities (3.3) for $k = 1, 2, \ldots, (j-1)$

with (3.1) and recalling the definition of $u_\theta(x,t)$ we obtain

$$(3.4)\qquad u_\theta(x,t)=\int_0^b N(x,t;\xi,0)h(\xi)\,d\xi-\int_0^t N(x,t;0,\tau)g\big(u_\theta(0,\tau),\tau\big)\,d\tau+$$
$$+\sum_{k=1}^{j-1}\int_{(k-1)\theta}^{k\theta} N(x,t;x_k,\tau)u_x^k(x_k,\tau)\,d\tau+\int_{(j-1)\theta}^{t} N(x,t;x_j,\tau)u_x^j(x_j,\tau)\,d\tau\,.$$

LEMMA 3.1. For each θ the following estimates are valid

(a) $\|u_\theta(0,\cdot)\|_T\leqslant\left[Hb+\dfrac{2G_2\sqrt{T}}{\sqrt{\pi}}\right]\exp\,[2G_1\sqrt{T}/\sqrt{\pi}]\equiv C_0\,,$

(b) $\|u_\theta\|_{\infty,S}\leqslant C_0$.

PROOF. By virtue of the maximum principle $u_\theta\geqslant 0$ in $\mathfrak{D}_{s_\theta}$ and $u_x^j(x_j,t)\leqslant 0$, $t\in\big((j-1)\theta, j\theta\big]$, therefore dropping the non-positive terms on the right hand side of (3.4) and letting $x\to 0$ we obtain

$$0\leqslant u_\theta(0,t)\leqslant\int_0^b 2\Gamma(0,t;\xi,0)h(\xi)\,d\xi-\frac{1}{\sqrt{\pi}}\int_0^t\frac{g\big(u_\theta(0,\tau),\tau\big)}{\sqrt{t-\tau}}\,d\tau\leqslant$$
$$\leqslant\frac{Hb}{\sqrt{\pi}}\int_0^b\frac{1}{\sqrt{t}}\exp\,[-\xi^2/4t]\,d\xi+\frac{G_1}{\sqrt{\pi}}\int_0^t\frac{u_\theta(0,\tau)}{\sqrt{t-\tau}}\,d\tau+\frac{2G_2}{\sqrt{\pi}}\sqrt{T}\,.$$

Statement (a) is now a consequence of standard calculations and Gronwall's inequality.

Statement (b) follows from the maximum principle applied recursively to $(\mathfrak{P}_j)$.

LEMMA 3.2. For each $j=1,2,\ldots,n$

(a) $|u_x^j(x_j,t)|\leqslant\tilde{H}\exp\,[\overline{T}]+8G\overline{T}\exp\,[\overline{T}]\equiv C_1,\ \overline{T}=T/\sqrt{\pi}b^2$

(b) $\left|\dfrac{\partial}{\partial x}u_\theta(x,t)\right|\leqslant C_1,\ (x,t)\in\mathfrak{D}_{s_\theta},$

where $G=G_1C_0+G_2$, and $\tilde{H}=\max\{H,G\}$.

PROOF. We employ an induction argument by making use of formulae (3.2). First we prove that if for some $j = 1, 2, \ldots, (n-1)$ we have

$$|u_x^j(x, j\theta)| \leqslant P\,, \qquad 0 < x < x_j$$

for some positive constant P, then

$$|u_x^{j+1}(x_{j+1}, t)| \leqslant P \exp[\theta/\sqrt{\pi} b^2] + 8G \frac{\theta}{\sqrt{\pi} b^2} \exp[\theta/\sqrt{\pi} b^2]\,, \qquad t \in (j\theta, (j+1)\theta)] \tag{3.5}$$

where

$$\max_{(0,T]} |g(u_\theta(0, t), t)| \leqslant G_1 C_0 + G_2 \equiv G\,.$$

Consider (3.2) written for the integer $j+1$

$$\tfrac{1}{2} u_x^{j+1}(x_{j+1}, t) = I_1 + I_2 + I_3 \tag{3.6}$$

and estimate the integrals I_i, $i = 1, 2, 3$ separately as follows.

$$|I_1| = \left| \int_0^{x_{j+1}} G(x_{j+1}, t; \xi, j\theta) u_x^{j+1}(\xi, j\theta)\, d\xi \right| = \text{by definition of } (\mathcal{T}_j)$$

$$= \left| \int_0^{x_j} G(x_{j+1}, t; \xi, j\theta) u_x^j(\xi, j\theta)\, d\xi \right| \leqslant P \int_0^{x_j} |G(x_{j+1}, t; \xi, j\theta)|\, d\xi$$

$$\leqslant \frac{P}{\sqrt{\pi}} \int_0^\infty \exp[-\eta^2]\, d\eta = \frac{P}{2}\,.$$

To estimate I_2, I_3 we recall the following elementary estimates on N_x

$$|N_x(x_{j+1}, t; 0, \tau)| \leqslant \frac{4}{\sqrt{\pi} b^2}\,, \qquad (x_{j+1} \geqslant b > 0)$$

$$|N_x(x_{j+1}, t; x_{j+1}, \tau)| \leqslant \frac{1}{2\sqrt{\pi} b^2}\,.$$

Therefore

$$|I_2| \leqslant \frac{4G}{\sqrt{\pi b^2}}(t - j\theta) \leqslant 4G\frac{\theta}{\sqrt{\pi b^2}},$$

$$|I_3| \leqslant \frac{1}{2\sqrt{\pi b^2}}\int_{j\theta}^{t} |u_x^{j+1}(x_{j+1}, \tau)|\, d\tau .$$

Putting together these estimates as parts of (3.6) we have

$$|u_x^{j+1}(x_{j+1}, t)| \leqslant P + 8G\frac{\theta}{\sqrt{\pi b^2}} + \frac{1}{\sqrt{\pi b^2}}\int_{j\theta}^{t} |u_x^{j+1}(x_{j+1}, \tau)|\, d\tau$$

for all $t \in \big(j\theta, (j+1)\theta\big]$.

Consequently by Gronwall's inequality (3.5) follows at once.

Consider now the problem $(\mathcal{T}_j)$ $j = 1$. Since $x \to h(x)$ is Lipschitz continuous in $[0, b]$, $h'(x)$ exists for a.e. $x \in [0, b]$ and

$$\operatorname*{ess\,sup}_{[0,b]} |h'(x)| \leqslant H \leqslant \tilde{\mathrm{H}} = \max\{H, G\} .$$

Therefore by the previous argument

$$|u_x^1(x_1, t)| \leqslant \tilde{H}e^{\alpha} + 8G\alpha e^{\alpha}, \qquad t \in (0, \theta]$$

where for simplicity of notation we have set $\alpha = \theta/\sqrt{\pi b^2}$. The function $(x, t) \to u_x^1(x, t) \equiv v(x, t)$, will satisfy the Dirichlet problem

$$\begin{aligned}
&Lv = 0 && \text{in } R_1 \\
&v(0, t) = g\big(u_\theta(0, t), t\big)\,; && |g\big(u_\theta(0, t), t\big)| \leqslant G\,, \qquad t \in (0, \theta] \\
&|v(x_1, t)| \leqslant \tilde{H}e^{\alpha} + 8G\alpha e^{\alpha}\,, && t \in (0, \theta] \\
&v(x, 0) = h'(x)\,, && |h'(x)| \leqslant H \qquad \text{a.e. } x \in (0, b] .
\end{aligned}$$

Consequently by the maximum principle

$$|u_x^1(x, \theta)| \leqslant \tilde{H}e^{\alpha} + 8G\alpha e^{\alpha}$$

and by (3.5)

$$|u_x^2(x_2, t)| \leqslant \tilde{H}e^{2\alpha} + 8G\alpha e^{2\alpha} + 8G\alpha e^{\alpha}$$
$$\leqslant \tilde{H}e^{2\alpha} + 8G(2\alpha)\,e^{2\alpha}\,, \qquad t \in (\theta, 2\theta]\,.$$

Proceeding in this fashion we obtain

$$|u_x^j(x_j, t)| \leqslant \tilde{H}e^{j\alpha} + 8G(j\alpha)\,e^{j\alpha}\,.$$

Now $j\alpha = (1/\sqrt{\pi}b^2)j\theta \leqslant \overline{T}$, and therefore the lemma is proved. Next we introduce the function $t \to \bar{s}_\theta(t)$ defined by

$$\text{(3.7)} \qquad \bar{s}_\theta(t) = x_j - u_x^j(x_j, j\theta)(t - (j-1)\theta)\,, \qquad t \in ((j-1)\theta, j\theta]\,.$$

For $t = j\theta$, $\bar{s}_\theta(t) = x_{j+1}$, $\bar{s}_\theta(0) = b$, so that the graph of $\bar{s}_\theta(\cdot)$ is obtained by connecting the points

$$(b, 0),\ (x_2, \theta), \ldots, \big(x_n, (n-1)\theta\big) \qquad \text{for } t \in \big(0, (n-1)\theta\big]\,,$$

and by the graph of (3.7) for $t \in [(n-1)\theta, T]$. The points $\big(x_j, (j-1)\theta\big)$, $j = 1, 2, \ldots, n$, are the lower vertices at the right side of the R_j's.

By Lemma 3.2

$$b \leqslant \bar{s}_\theta(t) \leqslant b + C_1 T\,, \qquad t \in [0, T]$$

therefore the sequence $\{\bar{s}_\theta(\cdot)\}$ is equibounded and equilipschitz, so that by Ascoli-Arzelà theorem a subsequence, relabeled with θ, converges uniformly to some nondecreasing, Lipschitz continuous curve $t \to s^*(t)$, with Lipschitz constant bounded by C_1. Since $\|s_\theta - \bar{s}_\theta\|_T \leqslant C_1\theta$, also $s_\theta(t)$ converges uniformly to $s^*(t)$.

Let D_{s^*} be the domain defined by

$$D_{s^*} \equiv \{0 < x < s^*(t)\} \times (0, T]\,,$$

and let u^* be the unique solution of

$$(\mathcal{T}) \qquad \begin{cases} Lu^* = 0 \ \text{ in } D_{s^*}\,, & \\ u_x^*(0, t) = g\big(u^*(0, t), t\big)\,, & t \in (0, T]\,, \\ u^*(x, 0) = h(x)\,, & x \in (0, b]\,, \\ u^*\big(s(t), t\big) = 0\,, & t \in (0, T]\,. \end{cases}$$

We will show that $u_\theta(x, t) \to u^*(x, t)$ uniformly in S and that the pair (u^*, s^*) so obtained is actually the unique solution of $(\mathcal{S}\mathcal{T})$ in the introduction.

We remark that as a consequence, in view of the uniqueness for $(\mathcal{S}\mathcal{T})$, the selection of subsequences is superfluous.

The following lemma will be needed.

LEMMA 3.3. Let $(x, t) \to v(x, t)$ be the unique solution of

$$\begin{aligned} Lv &= 0 && \text{in } D_{s^*}, \\ v_x(0, t) &= g\big(u_\theta(0, t), t\big), && t \in (0, T], \\ v(x, 0) &= h(x), && x \in (0, b], \\ v\big(s^*(t), t\big) &= 0, && t \in (0, T]. \end{aligned}$$

Then

$$0 \leqslant v(x, t) \leqslant C_1\big(s^*(t) - x\big), \qquad (x, t) \in D_{s^*}.$$

PROOF. The lemma is proved by standard barrier techniques and the maximum principle [10].

4. Convergence of the scheme.

LEMMA 4.1. $u_\theta(x, t) \to u^*(x, t)$ uniformly in S, as $\theta \to 0$.

PROOF. By the triangle inequality

$$\|u_\theta - u^*\|_{\infty,S} \leqslant \|w_1\|_{\infty,S} + \|w_2\|_{\infty,S}$$

where $w_1(x, t) = u_\theta(x, t) - v(x, t)$ and $w_2(x, t) = v(x, t) - u^*(x, t)$, and v is defined in Lemma 3.3.

Set

$$\alpha(t) = \min\{s_\theta(t), s^*(t)\}; \qquad \beta(t) = \max\{s_\theta(t), s^*(t)\},$$

$$\delta(t) = \beta(t) - \alpha(t), \qquad t \in [0, T].$$

We already know that $\delta(t) \to 0$ uniformly in $[0, T]$, as $\theta \to 0$.

Consider the rectangle R_j, $1 \leqslant j < n$. We claim that if

$$|w_1(x,(j-1)\theta)| \leqslant C_1 \|\delta\|_{(j-1)\theta}, \qquad x \in (0,\infty),$$

then

$$|w_1(x,t)| \leqslant C_1\|\delta\|_t, \qquad (x,t) \in (0,\infty) \times \big((j-1)\theta, j\theta\big].$$

If for $t \in [(j-1)\theta, j\theta]$, $x_j \leqslant s^*(t)$ then w_1 solves the problem

$$\begin{aligned}
Lw_1 &= 0 && \text{in } \{0 < x < x_j\} \times \big((j-1)\theta, j\theta\big],\\
w_{1_x}(0,t) &= 0\,, && t \in \big((j-1)\theta, j\theta\big],\\
|w_1(x,(j-1)\theta)| &\leqslant C_1\|\delta\|_{(j-1)\theta}\,, && x \in (0, x_j],\\
w_1(x_j,t) &= -v(x_j,t)\,, && t \in \big((j-1)\theta, j\theta\big].
\end{aligned}$$

Hence $|w_1(x,t)| \leqslant \max\,\{C_1\|\delta\|_{(j-1)\theta}, \max\limits_{[(j-1)\theta, j\theta]} v(x_j,t)\}$, $(x,t) \in R_j$. By Lemma 3.3 we obtain

$$|w_1(x,t)| \leqslant C_1\|\delta\|_t, \qquad (x,t) \in [0,\infty) \times [(j-1)\theta, j\theta].$$

If for $t \in [(j-1)\theta, j\theta]$, $x_j \geqslant s^*(t)$, then w_1 solves the problem

$$\begin{aligned}
Lw_1 &= 0 && \text{in } \{0 < x < s^*(t)\} \times \big((j-1)\theta, j\theta\big],\\
w_{1_x}(0,t) &= 0\,, && t \in \big((j-1)\theta, j\theta\big],\\
|w_1(x,(j-1)\theta)| &\leqslant C_1\|\delta\|_{(j-1)\theta}\,, && 0 < x \leqslant s^*[(j-1)\theta],\\
w_1(s^*(t),t) &= u_\theta(s^*(t),t)\,, && t \in \big((j-1)\theta, j\theta\big].
\end{aligned}$$

By Lemma 3.2 we have $0 \leqslant u_\theta(x,t) \leqslant C_1(x_j - x)$, $(x,t) \in R_j$ and therefore

$$|w_1(x,t)| \leqslant C_1\|\delta\|_t, \qquad (x,t) \in (0,\infty) \times \big((j-1)\theta, j\theta\big].$$

If $t^* \in \big((j-1)\theta, j\theta\big)$ such that $x_j = s^*(t^*)$, then we repeat analogous arguments in the domains so determined.

Now since for $t = 0$, $w_1(x,0) = 0$, $x \in (0,\infty)$, an inductive argument gives

$$|w_1(x,t)| \leqslant C_1\|\delta\|_t, \qquad x \in (0,\infty), \qquad t \in [0,T].$$

As for w_2, since it solves the problem

$$\begin{aligned} Lw_2 &= 0 && \text{in } D_{s^*} \\ w_{2x}(0,t) &= g(u_\theta(0,t),t) - g(u^*(0,t),t)\,, && t\in(0,T] \\ w_2(x,0) &= 0\,, && x\in(0,b] \\ w_2(s^*(t),t) &= 0\,, && t\in(0,T] \end{aligned}$$

it can be dominated [2, 3] by the function

$$\tilde{w}_2(x,t) = \int_0^t N(x,t;0,\tau)|g(u_\theta(0,\tau),\tau) - g(u^*(0,\tau),\tau)|d\tau$$

the unique solution of

$$\begin{aligned} L\tilde{w}_2 &= 0 && \text{in } S, \\ \tilde{w}_{2x}(0,t) &= -\,|g(u_\theta(0,t),t) - g(u^*(0,t),t)|\,, && t\in(0,T], \\ \tilde{w}_2(x,0) &= 0\,, && x\in(0,\infty). \end{aligned}$$

Hence

$$|w_2(x,t)| \leqslant G_1\int_0^t N(x,t;0,\tau)|u_\theta(0,\tau) - u^*(0,\tau)|d\tau\,.$$

We deduce that

$$\begin{aligned} |u_\theta(x,t) - u^*(x,t)| &\leqslant |w_1(x,t)| + G_1\frac{1}{\sqrt{\pi}}\int_0^t \frac{|u_\theta(0,\tau) - u^*(0,\tau)|}{\sqrt{t-\tau}}\,d\tau \\ &\leqslant C_1\|\delta\|_t + \frac{G_1}{\sqrt{\pi}}\int_0^t \frac{\sup\limits_{x\in[0,\infty)}|u_\theta(x,\tau) - u(x,\tau)|}{\sqrt{t-\tau}}\,d\tau\,. \end{aligned}$$

And by Gronwall's inequality

$$\sup_{x\in[0,\infty)}|u_\theta(x,t) - u(x,t)| \leqslant C_1\|\delta\|_t \exp\left[\frac{2G_1}{\sqrt{\pi}}\sqrt{T}\right] = C_2\|\delta\|_t \tag{4.1}$$

for all $t\in[0,T]$. The lemma is proved.

LEMMA 4.2. The pair (u^*, s^*) coincides with the unique solution of $(S\mathcal{T})$.

PROOF. The only thing that remains to be proved is the Stefan condition $u_x(s(t), t) = -\dot{s}(t)$, $t \in (0, T]$. Such a condition has been shown to be equivalent to the integral identity [2, page 85]

$$s(t) = b - \int_0^t g\big(u(0, \tau), \tau\big)\, d\tau + \int_0^b h(x)\, dx - \int_0^{s(t)} u(x, t)\, dx\,, \tag{4.2}$$

and hence it will be sufficient to prove that s^*, u^*, satisfy (4.2).

Integrating the equation $Lu^j = 0$ over R_j we obtain

$$\begin{aligned} -\int_{(j-1)\theta}^{j\theta} u_x^j(x_j, \tau)\, d\tau = -\int_{(j-1)\theta}^{j\theta} g\big(u^j(0, \tau), \tau\big)\, d\tau \,+ \\ + \int_0^{x_j} u^j\big(x, (j-1)\theta\big)\, dx - \int_0^{x_j} u^j(x, j\theta)\, dx\,. \end{aligned} \tag{4.3}$$

Also for $(x, t) \in R_p$, $1 \leqslant p \leqslant n$, integrate $Lu^p = 0$ over the rectangle $\{0 < x < x_p\} \times \big((p-1)\theta, t\big]$. It gives

$$\begin{aligned} -\int_{(p-1)\theta}^{t} u_x^p(x_p, \tau)\, d\tau = -\int_{(p-1)\theta}^{t} g\big(u^p(0, \tau), \tau\big)\, d\tau + \int_0^{x_p} u^p\big(x, (p-1)\theta\big)\, dx \,- \\ - \int_0^{x_p} u^p(x, t)\, dx\,. \end{aligned} \tag{4.4}$$

By the definition of $(\mathcal{T}_j)$ we have

$$\int_0^{x_{j+1}} u^{j+1}(x, j\theta)\, dx = \int_0^{x_j} u^j(x, j\theta)\, dx\,,$$

therefore adding the identities (4.3) for $j = 1, 2, \ldots, p-1$ and (4.4) we obtain

$$(4.5)\qquad -\sum_{j=1}^{p-1}\int_{(j-1)\theta}^{j\theta} u_x^j(x_j,\tau)\,d\tau - \int_{(p-1)\theta}^{t} u_x^p(x_p,\tau)\,d\tau = $$
$$= -\int_0^t g\big(u_\theta(0,\tau),\tau\big)\,d\tau + \int_0^b h(x)\,dx - \int_0^{x_p} u^p(x,t)\,dx\,.$$

We rewrite the left hand side of (4.5) as follows:

$$-\sum_{j=1}^{p-1}\int_{(j-1)\theta}^{j\theta} u_x^j(x_j,\tau)\,d\tau - \int_{(p-1)\theta}^{t} u_x^p(x_p,\tau)\,d\tau = -\sum_{j=1}^{p-1}\int_{(j-1)\theta}^{j\theta} u_x^j(x_j,j\theta)\,d\tau -$$
$$-\int_{(p-1)\theta}^{t} u_x^p(x_p,p\theta)\,d\tau - \sum_{j=1}^{p-1}\int_{(j-1)\theta}^{j\theta} e_\theta^j(\tau)\,d\tau - \int_{(p-1)\theta}^{t} e_\theta^p(\tau)\,d\tau\,,$$

where

$$e_\theta^i(t) = u_x^i(x_i,t) - u_x^i(x_i,i\theta)\,,\qquad (i-1)\theta < t \leqslant i\theta,\ i = 1, 2, \dots, n.$$

We observe that the numbers $-u_x^j(x_j, j\theta)$ are the slopes of the Lipschitz continuous polygonal $t \to \bar{s}_\theta(t)$ for $(j-1)\theta < t \leqslant j\theta$, consequently

$$-\sum_{j=1}^{p-1}\int_{(j-1)\theta}^{j\theta} u_x^j(x_j,j\theta)\,d\tau - \int_{(p-1)\theta}^{t} u_x^p(x_p,p\theta)\,d\tau = \int_0^t \frac{d}{d\tau}\bar{s}_\theta(\tau)\,d\tau = \bar{s}_\theta(t) - b\,.$$

Carrying this in (4.5) gives

$$(4.6)\qquad \bar{s}_\theta(t) = b - \int_0^t g\big(u_\theta(0,\tau),\tau\big)\,d\tau + \int_0^b h(x)\,dx - \int_0^{x_p} u_\theta(x,t)\,dx +$$
$$+\sum_{j=1}^{p-1}\int_{(j-1)\theta}^{j\theta} e_\theta^j(\tau)\,d\tau + \int_{(p-1)\theta}^{t} e_\theta^p(\tau)\,d\tau\,.$$

By virtue of Lemma 4.1 and the uniform convergence $\bar{s}_\theta(t) \to s^*(t)$, letting $\theta \to 0$ in (4.6) gives

$$
(4.7) \qquad s^*(t) = b - \int_0^t g\big(u^*(0, \tau), \tau\big)\, d\tau + \int_0^b h(x)\, dx - \int_0^{s^*(t)} u^*(x, t)\, dx +
$$

$$
+ \lim_{\theta \to 0} \left\{ \sum_{j=1}^{p-1} \int_{(j-1)\theta}^{j\theta} e_\theta^j(\tau)\, d\tau + \int_{(p-1)\theta}^{t} e_\theta^p(\tau)\, d\tau \right\}.
$$

Therefore the lemma will be proved if we show that the limit in (4.7) is zero.

In order to estimate the $e_\theta^j(\cdot)$ we will need a representation for $u_x^j(x_j, t)$, $t \in \big((j-1)\theta, j\theta\big]$.

Consider identity (3.3). By taking the derivative with respect to x and integrating by parts the first two integrals (the identity $N_x = -G_\xi$ is used) we obtain

$$
(4.8) \qquad \int_0^{x_k} G(x, t; \xi, k\theta)\, u_x^k(\xi, k\theta)\, d\xi -
$$

$$
- \int_0^{x_k} G\big(x, t; \xi, (k-1)\theta\big)\, u_x^k\big(\xi, (k-1)\theta\big)\, d\xi =
$$

$$
= - \int_{(k-1)\theta}^{k\theta} N_x(x, t; 0, \tau)\, g\big(u_\theta(0, \tau), \tau\big)\, d\tau + \int_{(k-1)\theta}^{k\theta} N_x(x, t; x_k, \tau)\, u_x^k(x_k, \tau)\, d\tau.
$$

Now by the construction of u^k, the second integral in (4.8) can be rewritten as

$$
- \int_0^{x_{k-1}} G\big(x, t; \xi, (k-1)\theta\big)\, u_x^{k-1}\big(\xi, (k-1)\theta\big)\, d\xi,
$$

therefore adding the identities in (4.8) for $k = 1, 2, \ldots, (j-1)$, changing the sign and evaluating the sum at $x = x_j$, gives

$$\int_0^b G(x_j, t; \xi, 0) h'(\xi)\, d\xi - \int_0^{x_{j-1}} G(x_j, t; \xi, (j-1)\theta) u_x^{j-1}(\xi, (j-1)\theta)\, d\xi = \tag{4.9}$$
$$= \int_0^{(j-1)\theta} N_x(x_j, t; 0, \tau) g(u_\theta(0, \tau), \tau)\, d\tau - \sum_{k=1}^{j-1} \int_{(k-1)\theta}^{k\theta} N_x(x_j, t; x_k, \tau) u_x^k(x_k, \tau)\, d\tau\,.$$

Consider now the representation (3.2). By the way u^j has been constructed, the first integral on the right hand side of (3.2) reads

$$\int_0^{x_{j-1}} G(x_j, t; \xi, (j-1)\theta) u_x^{j-1}(\xi, (j-1)\theta)\, d\xi\,.$$

Finally adding (3.2) and (4.9) gives

$$\frac{1}{2} u_x^j(x_j, t) = \int_0^b G(x_j, t; \xi, 0) h'(\xi)\, d\xi - \tag{4.10}$$
$$- \int_0^t N_x(x_j, t; 0, \tau) g(u_\theta(0, \tau), \tau)\, d\tau + \sum_{k=1}^{j-1} \int_{(k-1)\theta}^{k\theta} N_x(x_j, t; x_k, \tau) u_x^k(x_k, \tau)\, d\tau +$$
$$+ \int_{(j-1)\theta}^t N_x(x_j; t; x_j, \tau) u_x^j(x_j, \tau)\, d\tau\,, \qquad t \in ((j-1)\theta, j\theta]\,.$$

The error terms $e_\theta^j(t)$, $j = 2, 3, \ldots, (p-1)$, can now be written as

$$\frac{1}{2} e_\theta^j(t) = \frac{1}{2} [u_x^j(x_j, t) - u_x^j(x_j, j\theta)] =$$
$$= \int_0^b [G(x_j, t; \xi, 0) - G(x_j, j\theta; \xi, 0)] h'(\xi)\, d\xi -$$
$$- \int_0^t [N_x(x_j, t; 0, \tau) - N_x(x_j, j\theta; 0, \tau)] g(u_\theta(0, \tau), \tau)\, d\tau +$$

$$+\int_t^{j\theta} N_x(x_j, j\theta; 0, \tau) g(u_\theta(0, \tau), \tau)\, d\tau +$$

$$+\sum_{k=1}^{j-1} \int_{(k-1)\theta}^{k\theta} [N_x(x_j; t; x_k, \tau) - N_x(x_j, j\theta; x_k, \tau)] u_x^k(x_k, \tau)\, d\tau +$$

$$+\int_{(j-1)\theta}^{t} [N_x(x_j, t; x_j, \tau) - N_x(x_j, j\theta; x_j, \tau)] u_x^j(x_j, \tau)\, d\tau -$$

$$-\int_t^{j\theta} N_x(x_j, j\theta; x_j, \tau) u_x^j(x_j, \tau)\, d\tau = J_1 + J_2 + J_2' + J_3 + J_4 + J_4' .$$

By the mean value theorem and standard estimates on the exponential function we have

$$|G(x_j, t; \xi, 0) - G(x_j, j\theta; \xi, 0)| \leqslant \left(K_1 + \frac{K_2}{t^{\frac{3}{2}}}\right)\theta \leqslant K_1\theta + K_2(j-1)^{-\frac{3}{2}}\theta^{-\frac{1}{2}} .$$

Here and in what follows K_σ denote constants independent of j. Consequently

$$|J_1| \leqslant Hb(K_1\theta + K_2(j-1)^{-\frac{3}{2}}\theta^{-\frac{1}{2}}) .$$

As for J_2 analogous arguments yield the estimate

$$|N_x(x_j, t; 0, \tau) - N_x(x_j, j\theta; 0, \tau)| \leqslant K_3\theta ,$$

where we have used the fact that $x_j \geqslant b > 0$. Therefore J_2 can be estimated by

$$|J_2| \leqslant GK_3T\theta .$$

Let us now turn to J_3. First we note that for $k = 1, 2, \ldots, (j-2)$,

$$|N_x(x_j, t; x_k, \tau) - N_x(x_j, j\theta; x_k, \tau)| \leqslant (j\theta - t) \sup_{t \leqslant \varrho \leqslant j\theta} |N_{xt}(x_j, \varrho; x_k, \tau)| \leqslant$$

$$\leqslant K_4\theta + K_5\theta \sup_{t \leqslant \varrho \leqslant j\theta} \frac{x_j - x_k}{(\varrho - \tau)^{\frac{5}{2}}} \exp\left[-\frac{(x_j - x_k)^2}{4(\varrho - \tau)}\right] .$$

Now since $(x_k, k\theta)$, $(x_j, (j-1)\theta)$ belong to the maximal extension of the graph $(t, s_\theta(t))$, we have $(x_j - x_k) \leqslant 2C_1(\varrho - \tau)$, for $k = 1, 2, \ldots, (j-2)$, and hence

$$|J_3| \leqslant \sum_{k=1}^{j-2} \int_{(k-1)\theta}^{k\theta} |N_x(x_j, t; x_k, \tau) - N_x(x_j, j\theta; x_k, \tau)||u_x^k(x_k, \tau)|\, d\tau +$$

$$+ \int_{(j-2)\theta}^{(j-1)\theta} |N_x(x_j, t; x_{j-1}, \tau) - N_x(x_j, j\theta; x_{j-1}, \tau)||u_x^{j-1}(x_{j-1}, \tau)|\, d\tau \leqslant$$

$$\leqslant C_1 K_4 \theta \int_0^{(j-2)\theta} d\tau + 2C_1^2 K_5 \theta \int_0^{(j-2)\theta} [(j-1)\theta - \tau]^{-\frac{3}{2}}\, d\tau +$$

$$+ C_1 \int_{(j-2)\theta}^{(j-1)\theta} |N_x(x_j, t; x_{j-1}, \tau) - N_x(x_j, j\theta; x_{j-1}, \tau)|\, d\tau\,.$$

In order to estimate the last integral recall that

$$|N_x(x_j, \varrho; x_{j-1}, \tau)| \leqslant K_6(\varrho - \tau)^{-\frac{1}{2}}\,, \qquad \varrho > \tau.$$

Substituting this in the estimate of J_3 we obtain after some algebra

$$|J_3| \leqslant K_7 \sqrt{\theta}\,.$$

Analogous considerations yield $|J_2'| + |J_4| + |J_4'| \leqslant K_8 \sqrt{\theta}$. Therefore the error terms $e_\theta^j(t)$ can be estimated by

$$|e_\theta^j(t)| \leqslant K_9 \sqrt{\theta} + K_{10}(j-1)^{-\frac{3}{2}} \theta^{-\frac{1}{2}}\,.$$

For the error E_θ in (4.6) we have

$$|E_\theta| \leqslant \sum_{j=1}^{p-1} \int_{(j-1)\theta}^{j\theta} |e_\theta^j(\tau)|\, d\tau + \int_{(p-1)\theta}^{t} |e_\theta^p(\tau)|\, d\tau \leqslant \int_0^{\theta} |e_\theta^1(\tau)|\, d\tau + \sum_{j=2}^{p} \int_{(j-1)\theta}^{j\theta} |e_\theta^j(\tau)|\, d\tau\,.$$

Now trivially $|e_\theta^1(\tau)|\leqslant 2C_1$. Therefore

$$|E_\theta|\leqslant 2C_1\theta + K_9\sqrt{\theta}\sum_{j=2}^{p}\theta + K_{10}\sqrt{\theta}\sum_{j=2}^{p}(j-1)^{-\frac{3}{2}}\,.$$

Since $\sum_{j=2}^{p}\theta < p\theta \leqslant T$, and the series $\sum_{j\geqslant 2}(j-1)^{-\frac{3}{2}}$ converges, we obtain

$$|E_\theta|\leqslant K_{11}\sqrt{\theta}\,. \tag{4.11}$$

This proves the lemma.

5. The error estimate.

The purpose of this section is to give an estimate of the speed of convergence of the approximate interface $s_\theta(t)$ to the true interface $s(t)$. In view of Lemma 4.1 and (4.1) this will also complete the proof of the theorem.

LEMMA 5.1. There exists a constant C depending upon H, b, G_1, G_2, T such that

$$\|s_\theta - s\|_T \leqslant C\sqrt{\theta}\,.$$

PROOF. Subtract (4.6) from (4.2) and use (4.11) to obtain

$$\begin{aligned}|s(t)-\bar{s}_\theta(t)|\leqslant \int_0^t |g(u(0,\tau),\tau)-g(u_\theta(0,\tau),\tau)|\,d\tau\, + \\ +\left|\int_0^{s(t)} u(x,t)\,dx - \int_0^{\bar{s}_\theta(t)} \bar{u}_\theta(x,t)\,dx\right| + \\ +\left|\int_0^{\bar{s}_\theta(t)} \bar{u}_\theta(x,t)\,dx - \int_0^{x_p} u_\theta(x,t)\,dx\right| + K_{11}\sqrt{\theta}\,,\end{aligned}$$

where $\bar{u}_\theta$ is the unique solution of the problem

$$\begin{aligned} &L\bar{u}_\theta = 0 && \text{in } \mathfrak{D}_{\bar{s}_\theta} \equiv \{0 < x < \bar{s}_\theta(t)\} \times (0, T], \\ &\bar{u}_{\theta x}(0, t) = g\big(u_\theta(0, t), t\big), && 0 < t \leqslant T, \\ &\bar{u}_\theta(x, 0) = h(x), && x \in (0, b], \\ &\bar{u}_\theta\big(\bar{s}_\theta(t), t\big) = 0, && 0 < t \leqslant T. \end{aligned}$$

On the basis of the maximum principle and standard barriers estimates [10,7], as in Lemma 3.3, we have

$$\begin{aligned} &0 \leqslant u(x, t) \leqslant C_1\big(s(t) - x\big), && (x, t) \in D_T, \\ &0 \leqslant \bar{u}_\theta(x, t) \leqslant C_1\big(\bar{s}_\theta(t) - x\big), && (x, t) \in \mathfrak{D}_{\bar{s}_\theta}. \end{aligned}$$

PROPOSITION 5.1. There exist constants B_1 and B_2 depending upon H, b, G_1, G_2, T, such that

$$\left|\int_0^{s(t)} u(x, t)\, dx - \int_0^{\bar{s}_\theta(t)} \bar{u}_\theta(x, t)\, dx\right| \leqslant B_1 \int_0^t \frac{\sup\limits_{0 \leqslant \varrho \leqslant \tau} |s(\varrho) - \bar{s}_\theta(\varrho)|}{\sqrt{t - \tau}}\, d\tau + B_2 \int_0^t \frac{\|\delta\|_\tau}{\sqrt{t - \tau}}\, d\tau .$$

PROOF OF PROPOSITION 5.1. Set

$$\bar{\alpha}(t) = \min\{s(t), \bar{s}_\theta(t)\}; \qquad \bar{\beta}(t) = \max\{s(t), \bar{s}_\theta(t)\},$$
$$\bar{\delta}(t) = \bar{\beta}(t) - \bar{\alpha}(t), \qquad t \in [0, T].$$

Obviously $\bar{\alpha}(\cdot)$, $\bar{\beta}(\cdot)$ are non-decreasing Lipschitz continuous functions with Lipschitz constant bounded by C_1. Then

$$\left|\int_0^{s(t)} u(x, t)\, dx - \int_0^{\bar{s}_\theta(t)} \bar{u}_\theta(x, t)\, dx\right| \leqslant \int_0^{\bar{\alpha}(t)} |u(x, t) - \bar{u}_\theta(x, t)|\, dx + \int_{\bar{\alpha}(t)}^{\bar{\beta}(t)} |y(x, t)|\, dx = I_1 + I_2$$

where

$$y(x, t) = \begin{cases} u(x, t) & \text{if } \bar{\alpha}(t) = \bar{s}_\theta(t) \\ \bar{u}_\theta(x, t) & \text{if } \bar{\alpha}(t) = s(t). \end{cases}$$

We dominate the integrand in I_1 by the sum $|v - \bar{u}_\theta| + |u - v| \leqslant v_1 + v_2$ where v is defined in Lemma 3.3 and v_1, v_2 are the solutions of the problems

$$(\mathcal{T}_{v_1}) \quad \begin{cases} Lv_1 = 0\,, & \{0 < x < \bar{\alpha}(t)\} \times \{0 < t \leqslant T\}\,, \\ v_{1x}(0, t) = 0\,, & 0 < t \leqslant T\,, \\ v_1(x, 0) = 0\,, & 0 < x \leqslant b\,, \\ v_1(\bar{\alpha}(t), t) = C_1 \bar{\delta}(t)\,, & 0 \leqslant t \leqslant T \end{cases}$$

and

$$(\mathcal{T}_{v_2}) \quad \begin{cases} Lv_2 = 0 \quad (0, \infty) \times (0, T]\,, \\ v_{2x}(0, t) = -\,|g(u(0, t), t) - g(u_\theta(0, t), t)|\,, \quad 0 < t \leqslant T\,, \\ v_2(x, 0) = 0\,, \quad x \in [0, \infty)\,. \end{cases}$$

As for v_2, it can be represented explicitly [9], by

$$v_2(x, t) = \int_0^t N(x, t; 0, \tau)|g(u(0, \tau), \tau) - g(u_\theta(0, \tau), \tau)|\,d\tau\,.$$

Therefore by virtue of the Lipschitz continuity of $g(\cdot, t)$ and (4.1) we obtain

$$v_2(x, t) \leqslant \frac{G_1}{\sqrt{\pi}}\, C_2 \int_0^t \frac{\|\delta\|_\tau}{\sqrt{t - \tau}}\, d\tau\,.$$

By an argument of [2] page 87, v_1 can be dominated by $z(x, t) + z(-x, t)$ where z solves

$$(\mathcal{T}_z) \quad \begin{cases} Lz = 0\,, & \{-\bar{\alpha}(t) < x < \infty\} \times \{0 < t \leqslant T\}\,, \\ z(-\bar{\alpha}(t), t) = C_1 \bar{\delta}(t)\,, & 0 < t \leqslant T\,, \\ z(x, 0) = 0\,, & -b \leqslant x < \infty\,. \end{cases}$$

Then there exists a constant K_1 depending only upon the data such that

$$z(x,t) \leqslant K_{12}\int_0^t \|\delta\|_\tau |\Gamma_x(x,t;-\bar{\alpha}(\tau),\tau)|\,d\tau .$$

Therefore

$$\int_0^{\bar{\alpha}(t)} v_1(x,t)\,dx \leqslant \int_{-\bar{\alpha}(t)}^{\bar{\alpha}(t)} z(x,t)\,dx \leqslant \int_{-\bar{\alpha}(t)}^{\infty} z(x,t)\,dx \leqslant K_{13}\int_0^t \frac{\|\bar{\delta}\|_\tau}{\sqrt{t-\tau}}\,d\tau .$$

The estimate of I_2 is done by exploiting again the methods of [2] page 87 and dominating the integrand by $\tilde{z}(x,t)$, the solution of

$$(\mathcal{T}_{\tilde{z}})\qquad \begin{cases} L\tilde{z} = 0 , & \{\bar{\alpha}(t) < x < \infty\}\times\{0 < t \leqslant T\} , \\ \tilde{z}(x,0) = 0 , & b \leqslant x < \infty , \\ \tilde{z}(\bar{\alpha}(t),t) = C_1\bar{\delta}(t) , & 0 < t \leqslant T . \end{cases}$$

It gives

$$I_2 \leqslant K_{14}\int_0^t \frac{\|\bar{\delta}\|_\tau}{\sqrt{t-\tau}}\,d\tau .$$

PROPOSITION 5.2. There exist a constant $\tilde{B}$ depending upon H, b, G_1, G_2, T such that

$$\left|\int_0^{\bar{s}_\theta(t)} \bar{u}_\theta(x,t)\,dx - \int_0^{x_p} u_\theta(x,t)\,dx\right| \leqslant \tilde{B}\theta .$$

PROOF OF PROPOSITION 5.2. For $t > 0$ fixed we have $\bar{s}_\theta(t) \geqslant x_p$. Then

$$\left|\int_0^{\bar{s}_\theta(t)} \bar{u}_\theta(x,t)\,dx - \int_0^{x_p} u_\theta(x,t)\,dx\right| \leqslant \int_0^{x_p} |\bar{u}_\theta(x,t) - u_\theta(x,t)|\,dx + \\ + \int_{x_p}^{\bar{s}_\theta(t)} \bar{u}_\theta(x,t)\,dx = J_1 + J_2 .$$

Standard barrier arguments [7, 8, 9] give

$$\sup_{(x,t)\in\mathfrak{D}_{\bar{s}_\theta}} |\bar{u}_\theta(x,t)| \leqslant C_1(b + C_1 T)\,.$$

On the other hand by the construction of $\bar{s}_\theta(\cdot)$, the distance $\bar{s}_\theta(t) - x_p$ is at most $C_1\theta$. Hence

$$J_2 \leqslant K_{15}\theta\,.$$

To estimate the integrand in J_1 we proceed as in Lemma 4.1. In fact, in the present situation the estimates are simpler since $\bar{u}_\theta - u_\theta$ has zero flux at the fixed face $x = 0$. We obtain the estimate analogous to (4.1)

$$|\bar{u}_\theta(x,t) - u_\theta(x,t)| \leqslant K_{16} \sup_{0\leqslant\tau\leqslant t} |\bar{s}_\theta(\tau) - s_\theta(\tau)|\,.$$

As remarked above for all $t \in [0, T]$ we have

$$\bar{s}_\theta(t) - s_\theta(t) \leqslant C_1\theta\,,$$

and hence the proposition is proved.

We now conclude the proof of Lemma 5.1. Notice that by virtue of the Lipschitz continuity of $g(\cdot, t)$ and (4.1) we have

$$\int_0^t |g(u(0,\tau),\tau) - g(u_\theta(0,\tau),\tau| \, d\tau \leqslant G_1 T^{\frac{1}{2}} \int_0^t \frac{|u(0,\tau) - u_\theta(0,\tau)|}{\sqrt{t-\tau}}\, d\tau \leqslant$$

$$\leqslant G_1 T^{\frac{1}{2}} C_2 \int_0^t \frac{\|\delta\|_\tau}{\sqrt{t-\tau}}\, d\tau\,.$$

Putting together the various estimates so obtained we see that for all $0 < t \leqslant T$ we have

$$|s(t) - \bar{s}_\theta(t)| \leqslant K_{18}\sqrt{\theta} + K_{19}\int_0^t \frac{\|\bar{\delta}\|_\tau}{\sqrt{t-\tau}}\, d\tau + K_{17}\int_0^t \frac{\|\delta\|_\tau}{\sqrt{t-\tau}}\, d\tau\,.$$

Now

$$|s(t) - \bar{s}_\theta(t)| \geqslant |s(t) - s_\theta(t)| - |s_\theta(t) - \bar{s}_\theta(t)|$$

and

$$\|\bar{\delta}\|_\tau = \sup_{0\leqslant\varrho\leqslant\tau} |s(\varrho) - \bar{s}_\theta(\varrho)| \leqslant \|\delta\|_\tau + \sup_{0\leqslant\varrho\leqslant\tau} |s_\theta(\varrho) - \bar{s}_\theta(\varrho)| \leqslant \|\delta\|_\tau + C_1\theta .$$

Consequently the above implies

$$\|\delta\|_t \leqslant K_{20}\sqrt{\theta} + K_{21}\int_0^t \frac{\|\delta\|_\tau}{\sqrt{t-\tau}}\, d\tau .$$

The proof of the lemma is concluded with an application of Gronwall's inequality.

6. Modifications of the scheme.

Consider the sequence of approximating problems $(\mathcal{T}_j)$ introduced in Section 2. The theorem remains true if we modify the flux condition on the fixed face $x = 0$ in $(\mathcal{T}_j)$ as follows

(6.i) Retarded flux.
We set $u_x^1(0, t) = g(h(0), t)$, $0 < t \leqslant \theta$ and for $j > 1$ we replace the flux condition in $(\mathcal{T}_j)$ with

$$u_x^j(0, t) = g\big(u^{j-1}(0, t-\theta), t\big), \qquad (j-1)\theta < t \leqslant j\theta .$$

(6.ii) Piecewise constant flux.
The modification consists in freezing the flux on $x = 0$ in the j-th rectangle at its value at the lower left corner.
Set $u_x^1(0, t) = g\big(h(0), 0\big)$, $0 < t \leqslant \theta$ and then for $j > 1$

$$u_x^j(0, t) = g\Big(u^{j-1}\big(0, (j-1)\theta\big), (j-1)\theta\Big), \qquad (j-1)\theta < t \leqslant j\theta .$$

The proof of the convergence and the error estimate for the scheme obtained with the above modifications, is carried out in essentially the same way as indicated in the previous sections. Some minor modifications are needed which we leave to the reader, (cf. [1]).

REFERENCES

[1] G. Borgioli - E. DiBenedetto - M. Ughi, *Stefan problems with non-linear boundary conditions: the polygonal method*, Zeit. Angew. Math. Mech., **58** (1978), pp. 539-546.

[2] J. R. Cannon - J. Douglas, *The stability of the boundary in a Stefan problem*, Ann. Scuola Norm. Sup. Pisa, **21** (1967), pp. 83-91.

[3] J. R. Cannon - C. D. Hill, *Existence, uniqueness, stability and monotone dependence in a Stefan problem for the heat equation*, J. Math. Mech., **17** (1967), pp. 1-20.

[4] O. A. Ladyzhenskaya - V. A. Solonnikov - N. N. Ural'tseva, *Linear and quasi-linear equations of parabolic type*, Amer. Math. Soc., Trans. of Math. Monographs, **23**, Providence, R.I., (1968).

[5] A. Datzeff, *Sur le problème linéaire de Stefan*, Annuaire Univ. Sofia Fac. Sci., **45** (1949), pp. 321-352.

[6] A. Datzeff, *Sur le problème linéaire de Stefan*, Memoire de Sciences Physiques, fasc. 69, Gauthiers-Villars, Paris (1970).

[7] A. Fasano - M. Primicerio, *Il problema di Stefan con condizioni al contorno non lineari*, Ann. Scuola Norm. Sup. Pisa, **26** (1972), pp. 711-737.

[8] A. Fasano - M. Primicerio, *Su un problema unidimensionale di diffusione in un mezzo a contorno mobile con condizioni ai limiti non lineari*, Ann. Mat. Pura Appl., IV ser., **93** (1972), pp. 333-357.

[9] A. Fasano - M. Primicerio, *La diffusione del calore in uno strato di spessore variabile in presenza di scambi termici con l'ambiente: deduzione di limitazioni a priori sulla temperatura e le sue derivate*, Rend. Sem. Mat. Univ. Padova, **50** (1973), pp. 269-330.

[10] A. Fasano - M. Primicerio, *Convergence of Hüber's method for heat conduction problems with change of phase*, Zeit. Angew. Math. Mech., **53** (1973), pp. 341-348.

[11] A. Fasano - M. Primicerio - F. Fontanella, *An efficient method for solving free-boundary diffusion problems*, Meccanica, **8** (1973), pp. 223-235.

[12] A. Friedman, *Partial differential equations of parabolic type*, Prentice Hall, Inc., 1964.

[13] M. Primicerio, *Problemi di diffusione a frontiera libera*, Boll. Un. Mat. Ital., (5), **18**-A (1981), pp. 11-68.

[14] J. C. W. Rogers - A. E. Berger - M. Ciment, *The alternating phase truncation method for numerical solutions of a Stefan problem*, SIAM J. Numer. Anal., **16** (1979), pp. 563-587.

[15] A. Hüber, *Hauptaufsätze über das Fortschreiten der Schmelzgrenze in einem linearen Leiter*, Zeit. Angew. Math. Mech., **19** (1939), pp. 1-21.

Manoscritto pervenuto in redazione il 26 settembre 1981.